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Conformal covariance  
 in  $2d$  conformal and integrable models,  
 in  $W$ -algebras  
 and in their supersymmetric extensions<sup>1</sup>

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**Abstract**

Conformal symmetry underlies the mathematical description of various two-dimensional integrable models (e.g. for their Lax representation, Poisson algebra, zero curvature representation,...) or of conformal models (for the anomalous Ward identities, operator product expansion, Krichever-Novikov algebra,...) and of  $W$ -algebras. Here, we review the construction of conformally covariant differential operators which allow to render the conformal covariance manifest. The  $N = 1$  and  $N = 2$  supersymmetric generalizations of these results are also indicated and it is shown that they involve nonstandard matrix formats of Lie superalgebras.

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## Remembering Viktor Ogievetsky

Since the SQS'99 seminar is part of a series of International Seminars dedicated to the memory of V. Ogievetsky, I would like to recall briefly this great physicist and personality. I had the chance to meet Viktor when he came to CERN around 1989, and also later on, when he was on visit at the University of Munich. The memories that I am keeping of these encounters are those of a very warm, kind and generous person, those of a physicist who was enthusiastic about his work and always encouraging his colleagues, especially the younger ones, in their endeavor. Thus, I believe Viktor Ogievetsky should not only be remembered in science through his important contributions to physics, but also through his general attitude towards research and all those involved in it.

## 1 Introduction

Conformal covariance is essential for the global formulation of scale invariant theories (conformal models) on compact Riemann surfaces of any genus. It is at the heart of  $W$ -algebras which are non-linear generalizations of the two-dimensional conformal algebra, i.e. of the Virasoro algebra. Moreover, as was realized in the eighties and nineties, conformal symmetry manifests itself in several respects in two-dimensional integrable models like the KdV or Boussinesq equations.

By taking into account the underlying symmetries of a given theory, one usually gains a better understanding of this theory [1]. From a practical point of view, these symmetries generally provide a useful tool for determining solutions or for checking results within a given theory.

Within the aforementioned theories and models, conformal symmetry manifests itself by the occurrence of conformally covariant differential operators in the time evolution equations or in the structure relations. In the present notes, we briefly review the definition and construction of these operators and of their supersymmetric extensions.

In our write-up, we have tried to maintain the informal style of the oral presentation and therefore some results are only illustrated by the simplest examples. For more details, we refer to the series of articles [2]-[6] and to the work cited therein. (Among the latter, we explicitly mention references [7, 8, 9] which represent the basis for some parts of [2]-[6].) In reference [6], we illustrate how conformally covariant operators enter the physical models we

mentioned and we show how they constrain, or largely determine, the form of some of these theories.

## 2 Geometric framework

### 2.1 Basic definitions

The arena we will work on, is a *Riemann surface*  $\Sigma$ , i.e. a connected, topological 2-manifold which is equipped with a *complex structure* (or equivalently, a real, smooth, connected and oriented 2-manifold which is equipped with a *conformal class of metrics*) [10]. Roughly speaking, this means that any two systems of local complex coordinates, say  $z$  and  $z'$ , are related by a conformal coordinate transformation,

$$z \xrightarrow{\text{conf.}} z'(z).$$

In the following, we will use the notation  $\partial \equiv \frac{\partial}{\partial z}$  and we will denote the complex conjugate of  $z$  by  $\bar{z}$ . Moreover, we assume that the considered Riemann surfaces are *compact* so that they are characterized by their *genus*  $g \geq 0$ .

A *conformal* (or *primary*) *field* of *weight*  $k \in \mathbf{Z}/2$  on the Riemann surface  $\Sigma$  is a collection  $\{c(z, \bar{z})\}$  of local complex-valued functions on  $\Sigma$  (one for each coordinate system  $(z, \bar{z})$ ), transforming according to

$$c'(z', \bar{z}') = (\partial z')^{-k} c(z, \bar{z}) \quad (1)$$

under a conformal change of coordinates. Thus,  $c$  transforms linearly with a certain power of the Jacobian of the change of coordinates<sup>3</sup>. The space of conformal fields of weight  $k$  on  $\Sigma$  will be denoted by  $\mathcal{F}_k$ .

The *Schwarzian derivative* of a conformal change of coordinates  $z \rightarrow z'(z)$  is defined by

$$S(z'; z) = \partial^2 \ln \partial z' - \frac{1}{2} (\partial \ln \partial z')^2. \quad (2)$$

A *projective* (or *Schwarzian*) *connection* [10] on the Riemann surface  $\Sigma$  is a collection  $\{R(z, \bar{z})\}$  of local complex-valued functions on  $\Sigma$  with the properties

- (i)  $R$  is locally holomorphic, i.e.  $\partial_{\bar{z}} R = 0$ ,
- (ii)  $R$  transforms inhomogeneously with the Schwarzian derivative under a conformal change of coordinates  $z \rightarrow z'(z)$ :

$$R'(z') = (\partial z')^{-2} [R(z) - S(z'; z)]. \quad (3)$$

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<sup>3</sup>One can consider conformal fields which also transform with a certain power of  $\bar{\partial}z'$ , but we will not need them in the sequel.

Such connections exist globally on compact Riemann surfaces of any genus. From the physical point of view, the field  $R$  and its complex conjugate represent the components of the energy-momentum tensor in two-dimensional conformal field theory.

## 2.2 Projective coordinates

A change of local coordinates  $Z \rightarrow Z'(Z)$  which has the form

$$Z' = \frac{aZ + b}{cZ + d} \quad \text{with } a, b, c, d \in \mathbf{C} \quad \text{and } ad - bc = 1 , \quad (4)$$

is called a *projective* (or *Möbius* or *fractional linear*) *transformation*. We note that the associated Jacobian is given by

$$\partial_Z Z' = (cZ + d)^{-2} . \quad (5)$$

In the following, coordinates belonging to a projective atlas on the Riemann surface  $\Sigma$  will always be denoted by capital letters  $Z$  or  $Z'$ .

A *projective structure* on  $\Sigma$  is an atlas of local coordinates for which all coordinate transformations are projective. Every Riemann surface admits such a structure. As a matter of fact, there is a *one-to-one correspondence between projective structures and projective connections* [10], see section 3.3 below.

Let  $\Sigma$  be a compact Riemann surface with a given projective structure. Then, a *quasi-primary field* of weight  $k \in \mathbf{Z}/2$  on  $\Sigma$  is a collection  $\{C_k(Z, \bar{Z})\}$  of local complex-valued functions on  $\Sigma$  which transform linearly with the  $k$ -th power of the Jacobian (5) under a projective change of coordinates:

$$C'_k = (cZ + d)^{2k} C_k . \quad (6)$$

## 2.3 Covariant linear differential operators

Consider the local form of a linear, holomorphic differential operator of order  $n \in \mathbf{N}$ , which is defined on the Riemann surface  $\Sigma$ :

$$L^{(n)} = a_0^{(n)} \partial^n + a_1^{(n)} \partial^{n-1} + a_2^{(n)} \partial^{n-2} + \dots + a_n^{(n)} \quad \text{with } a_k^{(n)} = a_k^{(n)}(z) .$$

If the leading coefficient  $a_0^{(n)}$  does not vanish anywhere, we can divide it. Therefore, in the following, we will assume that  $a_0^{(n)} \equiv 1$ .

**Definition 2.1** A holomorphic,  $n$ -th order differential operator, which is locally given on the compact Riemann surface  $\Sigma$  by

$$L^{(n)} = \partial^n + a_1^{(n)}\partial^{n-1} + a_2^{(n)}\partial^{n-2} + \dots + a_n^{(n)} ,$$

is called conformally covariant if it maps conformal fields (of some weight  $p \in \mathbf{Z}/2$ ) to conformal fields:

$$L^{(n)} : \mathcal{F}_p \longrightarrow \mathcal{F}_{p+n} .$$

This requirement is equivalent to the one that  $L^{(n)}$  transforms according to the following operatorial relation under a conformal change of coordinates  $z \rightarrow z'(z)$ :

$$L^{(n)'} = (\partial z')^{-(p+n)} L^{(n)} (\partial z')^p . \quad (7)$$

According to the following result, the coefficient  $a_1^{(n)}$  of a conformally covariant operator can always be eliminated without destroying conformal covariance [6, 11].

**Theorem 2.1** Consider  $n \in \mathbf{N}^*$ . On a compact Riemann surface of genus  $g > 1$ , a conformally covariant operator  $L^{(n)}$  for which the coefficient  $a_1^{(n)}$  does not identically vanish, can only exist if it acts on conformal fields of weight

$$p = \frac{1-n}{2} .$$

In this case,  $a_1^{(n)}$  transforms linearly under a conformal change of coordinates  $z \rightarrow z'(z)$ ,

$$a_1^{(n)'} = (\partial z')^{-1} a_1^{(n)} , \quad (8)$$

and thereby one can consistently impose the vanishing of this coefficient. The transformation law of  $a_2^{(n)}$  then takes the simple form

$$a_2^{(n)'} = (\partial z')^{-2} \left[ a_2^{(n)} - k_n S(z'; z) \right] \quad \text{where } k_n = \frac{n(n^2 - 1)}{12} , \quad (9)$$

and where  $S$  denotes the Schwarzian derivative.

Accordingly, in the sequel, we will always consider conformally covariant operators which are normalized by  $a_0^{(n)} \equiv 1$ ,  $a_1^{(n)} \equiv 0$  and, for short, we will refer to these as CCO's:

**Definition 2.2** A CCO (conformally covariant operator) of order  $n$  on the compact Riemann surface  $\Sigma$  is a map

$$L^{(n)} : \mathcal{F}_{\frac{1-n}{2}} \longrightarrow \mathcal{F}_{\frac{1+n}{2}} \quad (10)$$

with the local expression

$$L^{(n)} = \partial^n + a_2^{(n)} \partial^{n-2} + \dots + a_n^{(n)} . \quad (11)$$

Here, the coefficients  $a_2^{(n)}, \dots, a_n^{(n)}$  are locally holomorphic functions on  $\Sigma$  and  $a_2^{(n)}$  is a multiple of a projective connection:

$$a_2^{(n)} = \frac{n(n^2 - 1)}{12} R .$$

The remaining coefficients  $a_3^{(n)}, \dots, a_n^{(n)}$  transform in a more complicated way than  $R$  under conformal changes of coordinates [8], so as to ensure the covariance (10).

### 3 CCO's

From the conceptual point of view, CCO's are best approached by starting from the special coordinate system where  $a_2^{(n)} = 0$  (i.e. by starting from projective coordinates  $Z$ ) and then going over to generic local coordinates  $z$  by a conformal transformation: the dependence of the operators on the projective structure then translates into a dependence on a projective connection. Therefore, we will first discuss operators on  $\Sigma$  which are covariant with respect to projective transformations.

#### 3.1 Möbius covariant operators

##### Class 1: Operators which only depend on the projective structure

The operator  $\partial_Z^n \equiv (\frac{\partial}{\partial Z})^n$  (where  $Z$  belongs to a projective atlas on  $\Sigma$ ) transforms homogeneously if it acts on a quasi-primary field of weight  $\frac{1-n}{2}$  [7]:

**Lemma 3.1 (Bol's lemma)** Consider a projective atlas on  $\Sigma$  with local changes of coordinates (4). If  $C_{\frac{1-n}{2}}(Z, \bar{Z})$  is a quasi-primary field on  $\Sigma$ , then  $\partial_Z^n C_{\frac{1-n}{2}}$  also is, i.e. it transforms according to

$$\left( \partial_Z^n C_{\frac{1-n}{2}} \right)' = (cZ + d)^{1+n} \partial_Z^n C_{\frac{1-n}{2}} \quad (n = 0, 1, 2, \dots) . \quad (12)$$

## Class 2: Operators which depend linearly on a quasi-primary field

For a given  $n \in \mathbb{N}$  with  $n \geq 3$ , we consider linear Möbius covariant operators  $M_{W_3}^{(n)}, \dots, M_{W_n}^{(n)}$  acting on quasi-primary fields of weight  $\frac{1-n}{2}$ . These operators do not only depend on the projective structure, but also, in a linear way, on quasi-primary fields  $W_3, \dots, W_n$ , respectively. Moreover, they are differential operators of lower order than  $\partial_Z^n$ .

Rather than giving a general formula for all of these operators (e.g. see [2]), we present their explicit expression for  $n = 5$ :

$$\begin{aligned} M_{W_5}^{(5)} &= W_5 \quad , \quad M_{W_4}^{(5)} = W_4 \partial_Z + \frac{1}{2}(\partial_Z W_4) \\ M_{W_3}^{(5)} &= W_3 \partial_Z^2 + (\partial_Z W_3) \partial_Z + \frac{2}{7}(\partial_Z^2 W_3) \quad . \end{aligned} \quad (13)$$

## 3.2 From projective to generic coordinates

Let us now go over from the projective coordinates  $Z$  to generic holomorphic coordinates  $z$  by a conformal transformation,

$$Z \xrightarrow{\text{conf.}} z \quad .$$

In doing so, a quasi-primary field  $C_k$  becomes a primary field  $c_k$ , both fields being related by

$$C_k(Z, \bar{Z}) = (\partial Z)^{-k} c_k(z, \bar{z}) \quad . \quad (14)$$

Moreover, a Möbius covariant operator becomes a CCO. To discuss this passage, we consider in turn the two classes of examples introduced above.

## 3.3 Class 1: Bol operators

When passing from the projective coordinates  $Z$  to the holomorphic coordinates  $z$  by a conformal transformation, the  $n$ -th order derivative  $\partial_Z^n$  acting on a quasi-primary field  $C_{\frac{1-n}{2}}$  becomes the  $n$ -th order *Bol operator* denoted by  $L_n$ :

$$\partial_Z^n C_{\frac{1-n}{2}} = (\partial Z)^{-\frac{1+n}{2}} L_n c_{\frac{1-n}{2}} \quad . \quad (15)$$

By substituting the relation (14) with  $k = \frac{1-n}{2}$  into equation (15), we obtain the following operatorial expression for the CCO  $L_n$ :

$$L_n = (\partial Z)^{\frac{1+n}{2}} \left( \frac{1}{\partial Z} \partial \right)^n (\partial Z)^{-\frac{1-n}{2}} \quad . \quad (16)$$

Thus, the Bol operator  $L_n$  represents the conformally covariant version of the differential operator  $\partial^n$ , the simplest examples being given by

$$\begin{aligned} L_0 &= 1 \\ L_1 &= \partial \\ L_2 &= \partial^2 + \frac{1}{2} R \\ L_3 &= \partial^3 + 2R\partial + (\partial R) \\ L_4 &= \partial^4 + 5R\partial^2 + 5(\partial R)\partial + \frac{3}{2} \left[ (\partial^2 R) + \frac{3}{2} R^2 \right] , \end{aligned} \quad (17)$$

where

$$R_{zz}(z) \equiv S(Z; z) . \quad (18)$$

This expression represents a projective connection, because it has the correct transformation properties thanks to the chain rule for the Schwarzian derivative. From this chain rule, it also follows that the definition (18) is not affected by a projective transformation of  $Z$ . Note that the quantity (18) is holomorphic since the change of coordinates  $z \rightarrow Z(z)$  has this property. Equation (18) expresses the one-to-one correspondence between projective structures and projective connections that we already mentioned.

The basic operator  $L_2$  (which is known as *Hill operator*) appears for instance in the Lax representation of the KdV equation while  $L_3$  appears in the Poisson brackets for the Virasoro algebra or in the conformal Ward identity [2].

### 3.4 Class 2: Operators depending linearly on conformal fields

Upon passage  $Z \rightarrow z$ , the quasi-primary field  $W_k$  becomes a primary field  $w_k$ , both fields being related by  $W_k = (\partial Z)^{-k} w_k$ . Moreover, the Möbius covariant operator  $M_{W_k}^{(n)}$  becomes a CCO  $M_{w_k}^{(n)}$  which depends linearly on  $w_k$  and which acts on  $\mathcal{F}_{\frac{1-n}{2}}$ . For instance, the  $n = 5$  operators (13) become

$$\begin{aligned} M_{w_5}^{(5)} &= w_5 , \quad M_{w_4}^{(5)} = w_4 \partial + \frac{1}{2} (\partial w_4) \\ M_{w_3}^{(5)} &= w_3 \left[ \partial^2 + 2R \right] + (\partial w_3) \partial + \frac{2}{7} \left[ \partial^2 - 3R \right] w_3 . \end{aligned} \quad (19)$$

### 3.5 Complete classification

Any CCO

$$L^{(n)} = \partial^n + a_2^{(n)}\partial^{n-2} + \dots + a_n^{(n)} \quad \text{with} \quad a_2^{(n)} = \frac{n(n^2 - 1)}{12} R$$

can be reparametrized in the following way in terms of the projective connection  $R$  and  $n - 2$  conformal fields  $w_3, \dots, w_n$ :

$$L^{(n)} = L_n + M_{w_3}^{(n)} + \dots + M_{w_n}^{(n)} . \quad (20)$$

The relation between the coefficients  $a_3^{(n)}, \dots, a_n^{(n)}$  and the conformal fields  $w_3, \dots, w_n$  is given by differential polynomials which involve  $R$  and this relation is invertible.

The parametrization (20) of  $L^{(n)}$  in terms of the energy-momentum tensor and some conformal fields is very helpful for the construction and formulation of  $W_n$ -algebras, see section 3.7 below.

### 3.6 Nonlinear conformally covariant operators

There exists a unique bilinear conformally covariant operator  $J(\cdot, \cdot)$ , the so-called *Gordan transvectant* [7, 8, 2]. Here, we only note that it encompasses the CCO's  $M_{w_k}^{(n)}$ :

$$M_{w_k}^{(n)} c \propto J(w_k, c) . \quad (21)$$

The bilinear operator  $J(\cdot, \cdot)$  as well as higher multilinear conformally covariant operators appear in the defining relations of  $W$ -algebras [3, 6].

### 3.7 Matrix representation of CCO's

The CCO's  $L_n$  and  $M_{w_k}^{(n)}$  admit a matrix representation which is related to the principal embedding of the Lie algebra  $sl(2)$  into  $sl(n)$  [8]. Since  $sl(2)$  is the Lie algebra of the Möbius group, this algebraic relationship which underlies the matrix representation of CCO's, reflects the fact that these covariant operators come from Möbius covariant ones. We will now illustrate the matrix representation for  $L^{(3)} = L_3 + M_{w_3}^{(3)} = L_3 + w_3$ .

Let us rewrite the scalar, conformally covariant differential equation

$$0 = L^{(3)} f_3 \equiv [\partial^3 + 2R\partial + (\partial R) + w_3] f_3 \quad \text{with } f_3 \in \mathcal{F}_{-\frac{1}{2}} \quad (22)$$

as a system of three first-order differential equations:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \partial & R & w_3 \\ -1 & \partial & R \\ 0 & -1 & \partial \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \iff \begin{cases} 0 = \partial f_1 + R f_2 + w_3 f_3 \\ f_1 = \partial f_2 + R f_3 \\ f_2 = \partial f_3 . \end{cases} \quad (23)$$

Substitution of the last two equations into the first one reproduces the scalar equation (22).

Equation (23) can also be written in the form

$$\vec{0} = (\partial - \mathcal{A}) \vec{F} \quad \text{with } \mathcal{A} = \begin{bmatrix} 0 & -R & -w_3 \\ 1 & 0 & -R \\ 0 & 1 & 0 \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} . \quad (24)$$

Here, the matrix  $\mathcal{A}$  can be viewed as the  $z$ -component of a two-dimensional gauge connection with values in the Lie algebra  $sl(3)$ . After supplementing  $\mathcal{A}$  with a  $\bar{z}$ -component, one can derive the  $W_3$ -algebra by imposing a *zero curvature condition* on the connection [9, 6].

## 4 $N = 1$ supersymmetry

### 4.1 General framework

The  $N = 1$  supersymmetric generalization of the previous results has been worked out in references [2, 3] (see also [12]) by using a superspace approach. We note that  $N = 1$  superspace is locally parametrized by complex coordinates  $z$  and  $\theta$  which are even and odd, respectively. The transition from ordinary space to superspace can be summarized as follows:

$$\left. \begin{array}{l} \text{Riemann surface} \\ z \\ \partial \\ \text{conformal transformation} \\ \text{conformal field } c_k \\ \text{projective connection } R_{zz}(z) \\ L_2 = \partial^2 + \frac{1}{2}R_{zz} \end{array} \right\} \rightarrow \left. \begin{array}{l} \text{super Riemann surface} \\ z, \theta \\ \partial, D \equiv \frac{\partial}{\partial \theta} + \theta \partial \quad (D^2 = \partial) \\ \text{superconformal transf. : } Dz' = \theta' D\theta' \\ \text{superconformal field : } C'_k = (D\theta')^{-k} C_k \\ \text{superprojective connection } \mathcal{R}_{z\theta}(z, \theta) \\ \mathcal{L}_1 = D^3 + \mathcal{R}_{z\theta} \end{array} \right. . \quad (25)$$

The odd superdifferential operator  $\mathcal{L}_1$  acts on a superconformal field  $C_{-1} \equiv C$ . By applying  $D$  to  $\mathcal{L}_1 C$  and subsequently projecting onto the lowest component

of the resulting superfield, we find

$$\begin{aligned} (D\mathcal{L}_1\mathcal{C})| &= \left[D^4 + (D\mathcal{R}_{z\theta})\right]\mathcal{C}| - \mathcal{R}_{z\theta}|(D\mathcal{C}_3)| \\ &= [\partial^2 + \frac{1}{2}R_{zz}]c + \rho_{z\theta}(D\mathcal{C})| , \end{aligned}$$

i.e. the basic Bol operator  $L_2$  plus a fermionic contribution.

## 4.2 Matrix representation of super CCO's

Let us rewrite the scalar, superconformally covariant differential equation

$$0 = \mathcal{L}_1 F_3 \equiv [D^3 + \mathcal{R}] F_3 \quad (26)$$

as a system of three first-order differential equations:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D & 0 & \mathcal{R} \\ -1 & D & 0 \\ 0 & -1 & D \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \iff \begin{cases} 0 = DF_1 + \mathcal{R}F_3 \\ F_1 = DF_2 \\ F_2 = DF_3 . \end{cases} \quad (27)$$

Analogously to the non supersymmetric theory, substitution of the last two equations into the first one reproduces the scalar equation (26).

If we now rewrite equation (27) in the form  $\vec{0} = (D - \mathcal{A})\vec{F}$ , we realize that the matrix  $\mathcal{A}$  belongs to the Lie superalgebra  $sl(2|1)$ . However, the graded matrix  $\mathcal{A}$  does not have the standard format which consists of arranging the even and odd matrix elements into blocks: this is an example of a *nonstandard matrix format*, to which we have referred as the *diagonal format* since there are alternatively even and odd diagonals [3, 5].

This and other possible nonstandard matrix formats have been studied in a systematic way in reference [5]. Although they are simply related to the standard format by a similarity transformation, they have many appealing features. Moreover, such formats naturally occur in various physical applications, e.g. in superconformal field theory, superintegrable models, for super  $W$ -algebras and quantum supergroups.

## 5 $N = 2$ supersymmetry

A  $N = 2$  super Riemann surface is locally parametrized by an even complex coordinate  $z$  and two odd complex coordinates  $\theta$  and  $\bar{\theta}$ . There is a new feature in  $N = 2$  superspace geometry which makes this theory considerably richer

and more complicated than the  $N = 1$  supersymmetric theory: the ‘square root’ of the translation generator  $\partial$  is not given by a single odd operator as in  $N = 1$  supersymmetry ( $D^2 = \partial$ ), but it involves two odd operators,

$$D = \frac{\partial}{\partial\theta} + \frac{1}{2}\bar{\theta}\partial \quad , \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \frac{1}{2}\theta\bar{\partial} \quad , \quad (28)$$

satisfying

$$\{D, \bar{D}\} = \partial \quad (29)$$

(and  $D^2 = 0 = \bar{D}^2$ ). Therefore, one has to deal with partial differential equations (involving  $D$  and  $\bar{D}$ ) rather than ordinary differential equations (only involving  $D$ ). Another aspect of the algebra  $\{D, \bar{D}\} = \partial$  consists of the fact that it introduces a  $U(1)$  symmetry into the theory: after projection from the super Riemann surface to the underlying ordinary Riemann surface, one thereby recovers  $U(1)$ -transformations in addition to the familiar conformal transformations. Henceforth, the Bol operators (17) acting on  $U(1)$ -neutral fields are to be generalized to conformally covariant operators acting on  $U(1)$ -charged fields. The latter as well as the original operators (17) arise from different types of  $N = 2$  super Bol operators which have been constructed and classified in reference [4]. For a particular class of them, the so-called ‘sandwich operators’ (relating the chiral and anti-chiral subspaces of superconformal fields), one can give a matrix representation. The results following from a zero curvature condition for the operator product expansions of the  $N = 2$  super  $W_3$ -algebra coincide with those obtained by other methods [13].

## 6 Concluding comments

In these notes, we have tried to give a short introduction to some mathematical notions which are quite useful for the study of many physically interesting models in two dimensions. While the appearance of conformal symmetry in conformal models or in their non-linear generalizations (related to  $W$ -algebras) is quite natural, the role of conformal invariance in integrable models is less clear and still a matter of current research (see references [14] and the contribution of M.Olshanetsky to this workshop).

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